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The acclamation consensus state and an associated ranking rule

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Abstract

The study of conditions, under which the existence of an "absolute" best winner can be assured, is a hot topic in the field of social choice. Unanimity is an evident example of a condition under which the winner is obvious. However, many more properties weaker than unanimity have been analysed in literature: the presence of a Condorcet winner, strong stochastic transitivity, the presence of a candidate that Borda dominates all other candidates, etc. Unfortunately, one could easily find a prominent ranking rule, for which the outcome does not agree with these relaxed conditions. In this study, we aim to identify a condition weaker than unanimity, but under which the social outcome is still obvious. This condition, defined as the conjunction of three properties already studied by the present authors and hereinafter referred to as acclamation, will be proven to be a meeting point for the most prominent ranking rules in social choice theory, and will be used for introducing an intuitively appealing ranking rule.

KEYWORDS

acclamation, monometric, ranking rule, social choice

1 | INTRODUCTION

The property of unanimity or Pareto efficiency, which appears as one of the three irreconcilable properties in Arrow's impossibility theorem,[1] requires that if every voter prefers a candidate to another one then so must the collective. The situation itself, in which all voters have the same preferences (loosely called unanimity also), is implicitly used in the definition of the most prominent ranking rules in the field of social choice. For instance, the method of Kemeny[2]

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could be understood as the search for the ranking that is the closest to becoming the unanimous ranking, and the Borda count[3] could be understood as the search for the candidate that is the closest to becoming the unanimous winner. Indeed, the search for unanimity is considered the standard procedure, not only for the aggregation of rankings,[4,5] but also for the aggregation of other types of mathematical objects. See, for instance, the use of penalty functions measuring the deviation from a 'consensus' element in the aggregation of real numbers.[6] Unfortunately, unanimity is no more than a utopian situation that rarely happens in real-life situations. Therefore, any method based on the search for unanimity strongly relies on the chosen measure of closeness to this unanimity. In order to reduce this importance of the chosen measure of closeness, the notion of unanimity needs to be softened.

A first example of a voting procedure in which unanimity was considered too strong, a requirement dates back to the times of Ancient Greece, where the Spartan Council's members were elected by the shouts of the attendees to the Assembly.[7] Although there was no full unanimity while choosing the elected members, the loud shouts of the people usually pointed to an obvious winner among all candidates. The fact of being the most applauded candidate was referred to as winning the election by acclamation. Another voting procedure, in which the notion of unanimity is clearly softened, is the method of Dodgson,[8] where the search for a unanimous winner is replaced by the search for a Condorcet winner.[9] For a more recent example, we refer to the search for the property of monotonicity proposed by Rademaker and De Baets.[10] Indeed, there exists a clear interest in the broadening of the notion of unanimity. Such broadened notions of unanimity are usually referred to as consensus states[11] and are considered a key element in the metric rationalisation of ranking rules,[11–14] which is the branch of social choice theory that aims to characterize a ranking rule as a procedure minimizing the distance to some consensus state.

In this study, we aim to identify the largest consensus state for which the winning ranking still is the obvious winner. For doing so, we recall three consensus states previously analysed by the present authors, namely, recursive monotonicity of the scorix,[15,16] monotonicity of the votrix[17] and monotonicity of the profile of rankings,[18] and introduce a stronger version of the latter one. The conjunction of these consensus states results in a new and natural consensus state that will be proven to be a meeting point for many prominent ranking rules found in the literature. Said consensus state will be hereinafter referred to as the *acclamation* consensus state. This term should be understood as a wink at the aforementioned Spartan voting system, for which the winner could be obvious, even in the absence of a unanimous winner. Moreover, we propose a Kemeny-like ranking rule based on the search for acclamation that will result in a more intuitive winner than the method of Kemeny itself.

The rest of the article is structured as follows: Section 2 is devoted to recalling some preliminary notions needed for the development of the article. In Section 3 we introduce the consensus state of acclamation and we prove it to be the cornerstone of social choice theory in absence of which the need of making a decision arises. A ranking rule based on the search for acclamation is proposed in Section 4. We end with some conclusions and open problems in Section 5.

2 | PRELIMINARIES

In this section, we recall several monotonicity-based consensus states that can be considered as cornerstones of social choice theory where families of ranking rules lead to the same social outcome. For more details concerning Sections 2.1, 2.2, and 2.3 we refer to Pérez-Fernández and co-workers.[15–18] In the considered problem setting, each of r voters expresses a strict total order

relation or ranking $\succ_j$ on a set $\mathscr{C} = \{a_1, ..., a_k\}$ of k candidates, that is, the asymmetric part of a total order relation $\succcurlyeq_j$ on $\mathscr{C}$. The set of all rankings on $\mathscr{C}$ is denoted by $\mathcal{L}(\mathscr{C})$ and the position at which candidate a_i is ranked in a ranking $\succ_j$ is denoted by $P_j(a_i)$ (we consider position 1 to be the best). Any list of r rankings is called a profile of rankings and is denoted by $\mathscr{R} = (\succ_j)_{j=1}^{r}$.

2.1 | Recursive monotonicity of the scorix

Each profile of rankings defines a matrix, henceforth called a scorix*, where each row represents a candidate in $\mathscr{C}$ and each column represents a position $\ell \in \{1, ..., k\}$. In this way, the element at the i-th row and ℓ-th column equals the number of times that the i-th candidate is ranked at the ℓ-th position.

Definition 1 (Pérez-Fernández et al[16]). Let $\mathscr{C}$ be a set of k candidates and $\mathscr{R}$ be the profile of r rankings on $\mathscr{C}$ given by the voters. The matrix $S \in \{0, 1, ..., r\}^{k \times k}$ defined as

$$S_{i\ell} = \#\{j \in \{1, ..., r\} | P_j(a_i) = \ell\},$$

for any $a_i \in \mathscr{C}$ and any $\ell \in \{1, ..., k\}$, is called the scorix induced by $\mathscr{R}$.

Borda[3] proposed to exploit these positions, at which every candidate is ranked, resulting in the introduction of the Borda ranking†.

Definition 2. Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings on $\mathscr{C}$ given by the voters and S be the scorix induced by $\mathscr{R}$. A ranking $\succ$ on $\mathscr{C}$ is called the Borda ranking if, for any $a_{i_1}, a_{i_2} \in \mathscr{C}$ such that $a_{i_1} \succ a_{i_2}$, it holds that

$$\sum_{\ell=1}^{k} (k - \ell) S_{i_1\ell} > \sum_{\ell=1}^{k} (k - \ell) S_{i_2\ell}.$$

A scorix is called monotone w.r.t. a ranking on the set of candidates if the vector of positions of each candidate dominates the vector of positions of all candidates ranked at a worse position in the given ranking.

Definition 3 (Pérez-Fernández et al[16]). Let $\mathscr{C}$ be a set of k candidates and r be the number of voters. A scorix S is said to be (strictly) monotone‡ w.r.t. a ranking $\succ$ on $\mathscr{C}$ if, for any $a_{i_1}, a_{i_2} \in \mathscr{C}$ such that $a_{i_1} \succ a_{i_2}$ and any $j \in \{1, ..., k\}$, it holds that

$$\sum_{\ell=1}^{j} S_{i_1\ell} \geq \sum_{\ell=1}^{j} S_{i_2\ell},$$

*The scorix is an old acquaintance of scholars in social choice theory,[19] usually considered either in the form of a matrix or in the form of a list of vectors (often called vectors of positions or vectors of ranks) corresponding to the different rows of the scorix.

†Note that some authors consider the Borda ranking to be a ranking with ties, thus being its existence assured. Here, because the Borda ranking is required to be strict, it is unique in case it exists.

‡For short, the word 'strictly' will be henceforth omitted. The same applies to, Definitions 5, 8, 10 and 11.

TABLE 1 Profile $\mathscr{R}$ of $r = 13$ rankings on $\mathscr{C} = \{a, b, c, d\}$

Frequency	Rankings on $\mathscr{C}$
3	$a \succ b \succ c \succ d$
2	$a \succ b \succ d \succ c$
2	$a \succ c \succ b \succ d$
2	$b \succ a \succ c \succ d$
1	$a \succ d \succ b \succ c$
1	$b \succ a \succ d \succ c$
1	$c \succ a \succ b \succ d$
1	$d \succ a \succ b \succ c$

and there exists at least one $j \in \{1, ..., k\}$ such that

$$\sum_{\ell=1}^{j} S_{i_1 \ell} > \sum_{\ell=1}^{j} S_{i_2 \ell}.$$

In the following example, the notions of scorix and monotonicity of a scorix are illustrated.

Example 1. Let $\mathscr{C} = \{a, b, c, d\}$ be a set of candidates and $\mathscr{R}$ be the profile of $r = 13$ rankings given in Table 1. Candidate a is ranked eight times at the first position and five times at the second position, while not being ranked at the third or fourth position. Therefore, the vector of positions of candidate a is $(8, 5, 0, 0)$. In the same way, the vector of positions of candidate b is $(3, 5, 5, 0)$, the vector of positions of candidate c is $(1, 2, 5, 5)$ and the vector of positions of candidate d is $(1, 1, 3, 8)$. The scorix induced by $\mathscr{R}$ is then given by:

$$S = \begin{pmatrix} 8 & 5 & 0 & 0 \\ 3 & 5 & 5 & 0 \\ 1 & 2 & 5 & 5 \\ 1 & 1 & 3 & 8 \end{pmatrix}.$$

In Figure 1, the scorix is represented on the Hasse diagram[20] of $\succ$ for the ranking $a \succ b \succ c \succ d$. Note that the vector of positions of candidate a dominates the vector of positions of candidates b, c, and d. Analogously, the vector of positions of candidate b dominates the vector of positions of candidates c and d, and the vector of positions of candidate c dominates the vector of positions of candidate d. Therefore, the scorix induced by the profile of rankings given in this example is monotone w.r.t. the ranking $a \succ b \succ c \succ d$.

For any non-empty subset $\mathscr{C}' \subseteq \mathscr{C}$, the restriction of a profile $\mathscr{R}$ of r rankings on $\mathscr{C}$ to $\mathscr{C}'$ is the profile $\mathscr{R}' = (\succ'_j)_{j=1}^{r}$ of r rankings on $\mathscr{C}'$ such that, for any $j \in \{1, ..., r\}$ and any $a_{i_1}, a_{i_2} \in \mathscr{C}'$, it holds that $a_{i_1} \succ'_j a_{i_2}$ if $a_{i_1} \succ_j a_{i_2}$. A matrix is said to be a sub-scorix of a scorix if it is the scorix associated with the restriction of the given profile of rankings to a subset of the set of candidates.

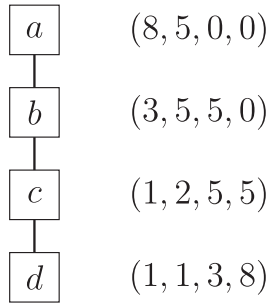

FIGURE 1 The scorix induced by the profile of rankings in Table 1 represented on the Hasse diagram of $\succ$ for the ranking $a \succ b \succ c \succ d$

Definition 4 (Pérez-Fernández and De Baets[15]). Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings on $\mathscr{C}$ given by the voters and S be the scorix induced by $\mathscr{R}$. For any non-empty subset $\mathscr{C}' \subseteq \mathscr{C}$, the scorix $S' \in \{0, 1, ..., r\}^{k' \times k'}$ (where $k' = |\mathscr{C}'|$) on $\mathscr{C}'$ induced by the restriction of $\mathscr{R}$ to $\mathscr{C}'$ is called the sub-scorix of S on $\mathscr{C}'$.

In the same way a scorix can be monotone, the respective sub-scorices can also be monotone. The monotonicity of all the sub-scorices of a scorix leads to a stronger type of monotonicity: recursive monotonicity of the scorix*.

Definition 5 (Pérez-Fernández and De Baets[15]). Let $\mathscr{C}$ be a set of k candidates and r be the number of voters. A scorix S is said to be (strictly) recursively monotone w.r.t. a ranking $\succ$ on $\mathscr{C}$ if, for any nonempty subset $\mathscr{C}' \subseteq \mathscr{C}$, the sub-scorix S' of S on $\mathscr{C}'$ is (strictly) monotone w.r.t. the restriction of $\succ$ to $\mathscr{C}'$.

In the following example, the notions of sub-scorix and recursive monotonicity of a scorix are illustrated.

Example 2. Let $\mathscr{C} = \{a, b, c, d\}$ be a set of candidates and $\mathscr{R}$ be the profile of rankings on $\mathscr{C}$ given in Example 1. The restriction of $\mathscr{R}$ to $\mathscr{C}' = \{a, b, c\}$ is the profile $\mathscr{R}'$ of rankings on $\mathscr{C}'$ listed in Table 2.

The scorix induced by $\mathscr{R}'$ is given by:

$$S' = \begin{pmatrix} 9 & 4 & 0 \\ 3 & 7 & 3 \\ 1 & 2 & 10 \end{pmatrix}.$$

Thus, S' is the sub-scorix of S on $\mathscr{C}'$.

In Figure 2 the scorix and all its sub-scorices are represented on the Hasse diagram of $\succ$ and its restriction to each subset of $\mathscr{C}$ (of cardinality greater than or equal to two) for the ranking $a \succ b \succ c \succ d$. Note that the scorix and all its sub-scorices are monotone w.r.t. the corresponding restriction of $a \succ b \succ c \succ d$. Therefore, the scorix given in this example is recursively monotone w.r.t. the ranking $a \succ b \succ c \succ d$.

*Recursive monotonicity of the scorix actually is a property of the profile of rankings and not of its scorix. This means that, given a scorix, it is not possible to identify whether or not it is recursively monotone without knowing the profile of rankings. For more details, we refer to Pérez-Fernández and De Baets[15] (Remark 3).

TABLE 2 Profile $\mathscr{R}'$ of $r = 13$ rankings on $\mathscr{C}' = \{a, b, c\}$

Frequency	Rankings on $\mathscr{C}$
7	$a \succ b \succ c$
3	$b \succ a \succ c$
2	$a \succ c \succ b$
1	$c \succ a \succ b$

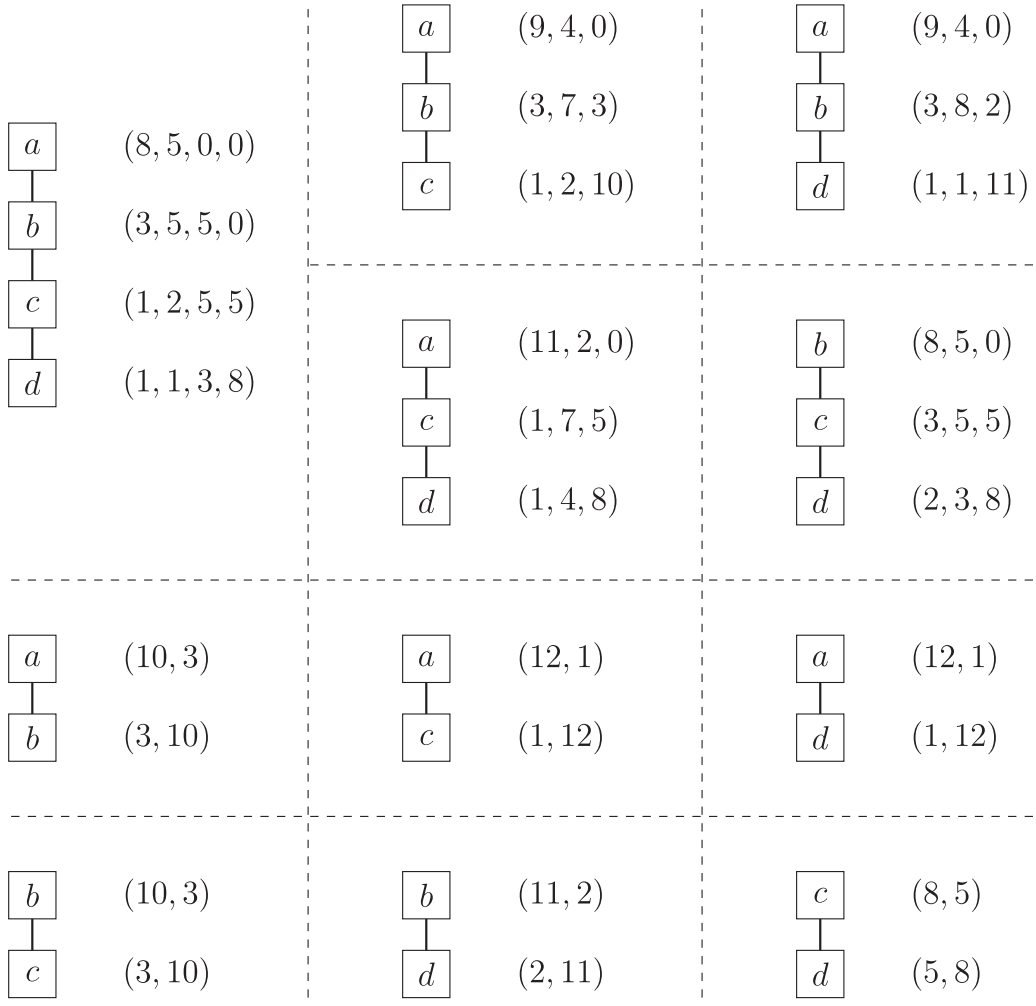

FIGURE 2 The scorix and sub-scorices induced by the profile of rankings in Table 1 represented on the Hasse diagram of $\succ$ and its restriction to each subset of $\mathscr{C} = \{a, b, c, d\}$ for the ranking $a \succ b \succ c \succ d$

From a totally different perspective than that advocated by Borda, Condorcet[9] proposed to exploit the pairwise comparisons of the candidates, resulting in the introduction of the Condorcet ranking*.

*The Condorcet ranking is unique in case it exists.

Definition 6. Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings on $\mathscr{C}$ given by the voters and S be the scorix induced by $\mathscr{R}$. A ranking $\succ$ on $\mathscr{C}$ is called the Condorcet ranking if any sub-scorix S' of S on a subset $\mathscr{C}'$ of $\mathscr{C}$ of cardinality two is monotone w.r.t. the restriction of $\succ$ to $\mathscr{C}'$.

Recursive monotonicity of the scorix obviously is a stronger property than monotonicity of the scorix and a weaker property than unanimity. Furthermore, it guarantees the existence of (and compliance with) both the Borda ranking and the Condorcet ranking.

Theorem 1 (Pérez-Fernández and De Baets[15]). *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of seven rankings on $\mathscr{C}$ given by the voters, S be the scorix induced by $\mathscr{R}$ and $\succ$ be a ranking on $\mathscr{C}$. The following statements hold:*

- **(i)** *If $\mathscr{R}$ is the unanimous profile of rankings where every voter expresses $\succ$, then S is recursively monotone w.r.t. $\succ$.*
- **(ii)** *If S is recursively monotone w.r.t. $\succ$, then S is monotone w.r.t. $\succ$.*
- **(iii)** *If S is recursively monotone w.r.t. $\succ$, then $\succ$ is the Condorcet ranking.*
- **(iv)** *If S is recursively monotone w.r.t. $\succ$, then every elimination method based on a scoring ranking rule defines a ranking (with ties) on the set of candidates that is linearly extended* by $\succ$.*
- **(v)** *If S is monotone w.r.t. $\succ$, then every scoring ranking rule defines a ranking (with ties) on the set of candidates that is linearly extended by $\succ$.*
- **(vi)** *If S is monotone w.r.t. $\succ$, then $\succ$ is the Borda ranking.*

2.2 | Monotonicity of the votrix

A common representation of votes based on pairwise information is the votrix†, a matrix where the element at the i-th row and j-th column equals the number of times that the i-th candidate has been preferred to the j-th candidate in the profile of rankings given by the voters.

Definition 7 (Pérez-Fernández et al[17]). Let $\mathscr{C}$ be a set of k candidates and $\mathscr{R}$ be the profile of r rankings on $\mathscr{C}$ given by the voters. The matrix $V \in \{0, 1, ..., r\}^{k \times k}$ defined as

$$V_{i_1 i_2} = \#\{j \in \{1, ..., r\} | a_{i_1} \succ_j a_{i_2}\},$$

for any $a_{i_1}, a_{i_2} \in \mathscr{C}$, is called the votrix induced by $\mathscr{R}$.

It is known that the Borda ranking and the Condorcet ranking can also be defined in terms of the votrix. Actually, the definition of the Condorcet ranking is commonly given by the following characterization, rather than by Definition 6.

*A ranking with ties is linearly extended by a ranking if any candidate that is ranked at a better position than another candidate in the ranking with ties is also ranked at a better position than this other candidate in the ranking.[20]

†The votrix is an old acquaintance of scholars in social choice theory[21,22] (for more details on representations of votes based on pairwise information, we refer to Pérez-Fernández and De Baets[23]), commonly referred to as the voting matrix.

Proposition 1 (Pérez-Fernández et al[17]). *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings on $\mathscr{C}$ given by the voters and V be the votrix induced by $\mathscr{R}$. The following two statements hold:*

(i) *A ranking $\succ$ on $\mathscr{C}$ is the Borda ranking if and only if, for any $a_{i_1}, a_{i_2} \in \mathscr{C}$ such that $a_{i_1} \succ a_{i_2}$, it holds that*

$$\sum_{a_\ell \in \mathscr{C}\setminus\{a_{i_1}\}} V_{i_1\ell} > \sum_{a_\ell \in \mathscr{C}\setminus\{a_{i_2}\}} V_{i_2\ell}\,.$$

(ii) *A ranking $\succ$ on $\mathscr{C}$ is the Condorcet ranking if and only if, for any $a_{i_1}, a_{i_2} \in \mathscr{C}$ such that $a_{i_1} \succ a_{i_2}$, it holds that*

$$V_{i_1 i_2} > V_{i_2 i_1}.$$

Any ranking $\succ$ on $\mathscr{C}$ naturally defines a strict partial order relation on $\mathscr{C}^2_{\neq} = \{(a_{i_1}, a_{i_2}) \in \mathscr{C}^2 | a_{i_1} \neq a_{i_2}\}$.

Proposition 2 (Rademaker and De Baets[10]). *Let $\mathscr{C}$ be a set of k candidates. A ranking $\succ$ on $\mathscr{C}$ induces the following strict partial order relation $\sqsupset_\succ$ on $\mathscr{C}^2_{\neq}$:*

$$\sqsupset_\succ = \left\{\left((a_{i_1}, a_{i_2}), (a_{i_3}, a_{i_4})\right) \in (\mathscr{C}^2_{\neq})^2 | (a_{i_1} \succcurlyeq a_{i_3}) \wedge (a_{i_4} \succcurlyeq a_{i_2}) \wedge (a_{i_1} \succ a_{i_3} \vee a_{i_4} \succ a_{i_2})\right\}.$$

Intuitively, given a ranking $\succ$, a couple of candidates is greater than another couple of candidates if the candidates in the first couple are more distant in $\succ$ than the candidates in the second couple. In Figure 3 the Hasse diagram of $\sqsupset_\succ$ for the ranking $a \succ b \succ c \succ d$ on the set of four candidates $\mathscr{C} = \{a, b, c, d\}$ is shown.

In case the values of the votrix decrease on the strict partial order relation $\sqsupset_\succ$ associated with the given ranking $\succ$ on $\mathscr{C}$ and, in addition, $\succ$ is the Condorcet ranking, the votrix is said to be monotone w.r.t. this ranking.

Definition 8 (Pérez-Fernández and De Baets[24]). Let $\mathscr{C}$ be a set of k candidates and r be the number of voters. A votrix V is said to be (strictly) monotone w.r.t. a ranking $\succ$ on $\mathscr{C}$ if, for any $(a_{i_1}, a_{i_2}), (a_{i_3}, a_{i_4}) \in \mathscr{C}^2_{\neq}$ such that $(a_{i_1}, a_{i_2}) \sqsupset_\succ (a_{i_3}, a_{i_4})$, it holds that

$$V_{i_1 i_2} \geq V_{i_3 i_4},$$

and, for any $(a_{i_1}, a_{i_2}) \in \mathscr{C}^2_{\neq}$ such that $a_{i_1} \succ a_{i_2}$, it holds that

$$V_{i_1 i_2} > V_{i_2 i_1}.$$

In the following example, the notions of votrix and monotonicity of a votrix are illustrated.

Example 3. Let $\mathscr{C} = \{a, b, c, d\}$ be a set of candidates and $\mathscr{R}$ be the profile of rankings given in Example 1. For instance, candidate a is ranked at a better position than

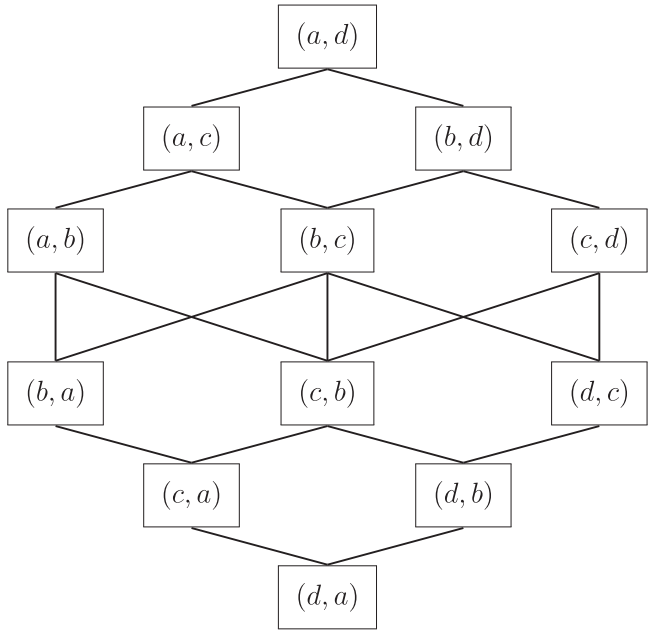

FIGURE 3 Hasse diagram of the order relation $\sqsupset_{\succ}$ for the ranking $a \succ b \succ c \succ d$

candidate b ten times. Therefore, the element at the first row and second column equals ten. The votrix induced by $\mathscr{R}$ is then given by:

$$V = \begin{pmatrix} 0 & 10 & 12 & 12 \\ 3 & 0 & 10 & 11 \\ 1 & 3 & 0 & 8 \\ 1 & 2 & 5 & 0 \end{pmatrix}.$$

In Figure 4 the votrix is represented on the Hasse diagram of $\sqsupset_{\succ}$ for the ranking $a \succ b \succ c \succ d$. Note that the values decrease when going from top to bottom in the Hasse diagram of $\sqsupset_{\succ}$. Therefore, the votrix induced by the profile of rankings given in this example is monotone w.r.t. the ranking $a \succ b \succ c \succ d$.

One could note that monotonicity of the votrix has a pairwise nature. Thus, a potentially-definable property of recursive monotonicity of the votrix would simply be equivalent to monotonicity of the votrix.

Monotonicity of the votrix obviously is a weaker property than unanimity. Furthermore, it guarantees the existence of (and compliance with) both the Borda ranking and the Condorcet ranking.

Theorem 2 (Pérez-Fernández et al[17]). *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings on $\mathscr{C}$ given by the voters, V be the votrix induced by $\mathscr{R}$ and $\succ$ be a ranking on $\mathscr{C}$. The following statements hold:*

(i) *If $\mathscr{R}$ is the unanimous profile of rankings where every voter expresses $\succ$, then V is monotone w.r.t. $\succ$.*

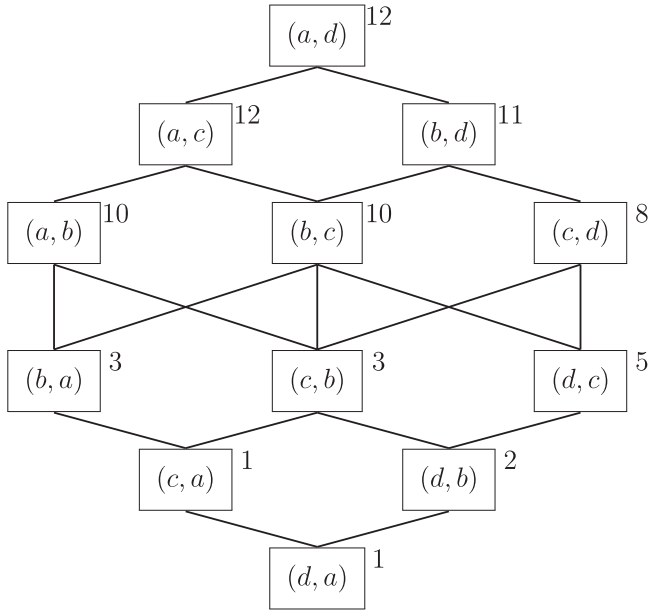

FIGURE 4 The votrix induced by the profile of rankings in Table 1 represented on the Hasse diagram of $\sqsupset_{\succ}$ for the ranking $a \succ b \succ c \succ d$

(ii) *If V is monotone w.r.t. $\succ$, then $\succ$ is the Borda ranking.*
(iii) *If V is monotone w.r.t. $\succ$, then $\succ$ is the Condorcet ranking.*

2.3 | Recursive monotonicity of the profile of rankings

Each ranking $\succ$ on $\mathscr{C}$ defines an order relation $\trianglerighteq_{\succ}$ on $\mathcal{L}(\mathscr{C})$ according to how far two rankings in $\mathcal{L}(\mathscr{C})$ are from $\succ$ in terms of reversals.

Definition 9 (Pérez-Fernández et al[18]). Let $\mathscr{C}$ be a set of k candidates. A ranking $\succ$ on $\mathscr{C}$ induces the following partial order relation $\trianglerighteq_{\succ}$ on $\mathcal{L}(\mathscr{C})$:

$$\trianglerighteq_{\succ} = \left\{(\succ', \succ'') \in \mathcal{L}(\mathscr{C})^2 \middle| \left(\forall\, (a_{i_1}, a_{i_2}) \in \mathscr{C}^2\right)\left(((a_{i_1} \succ a_{i_2}) \wedge (a_{i_1} \succ'' a_{i_2})) \Rightarrow (a_{i_1} \succ' a_{i_2})\right)\right\}.$$

Figure 5 displays the Hasse diagram of the order relation $\trianglerighteq_{\succ}$ for the ranking $a \succ b \succ c \succ d$ on the set of four candidates $\mathscr{C} = \{a, b, c, d\}$. Clearly, every ranking $\succ'$ is closer (in terms of reversals) to $\succ$ than $\succ''$ if it holds that $\succ' \trianglerighteq_{\succ} \succ''$.

Without taking the order of the voters into account, any profile of rankings is determined by the number of times that each ranking is expressed. The (absolute) frequency of the ranking $\succ$, that is, the number of voters that have expressed the ranking $\succ$ in the profile $\mathscr{R}$ of rankings, is denoted by $\mathbf{n}_{\mathscr{R}}(\succ)$. Note that it holds that

$$\sum_{\succ \in \mathscr{L}(\mathscr{C})} \mathbf{n}_{\mathscr{R}}(\succ) = r.$$

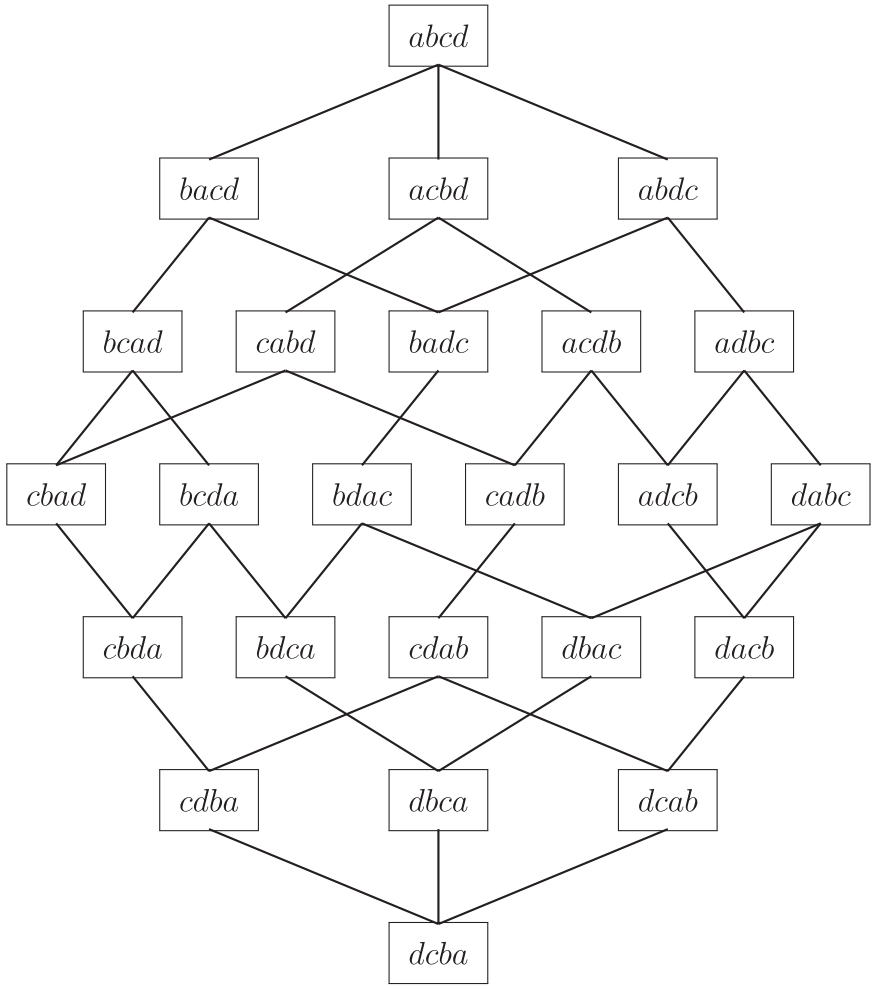

FIGURE 5 Hasse diagram of the order relation $\trianglerighteq_{\succ}$ for the ranking $a \succ b \succ c \succ d$, where $xyzt$ is a shorthand for $x \succ y \succ z \succ t$

Under the assumption that there exists a true ranking $\succ$ on $\mathscr{C}$, it seems natural that the frequencies of the rankings in the given profile of rankings decrease when going from top to bottom in the Hasse diagram of $\trianglerighteq_{\succ}$. A profile of rankings satisfying this property is said to be monotone w.r.t. the ranking $\succ$.

Definition 10 (Pérez-Fernández et al[18]). Let $\mathscr{C}$ be a set of k candidates and r be the number of voters. A profile $\mathscr{R}$ of r rankings on $\mathscr{C}$ is said to be (strictly) monotone w.r.t. a ranking $\succ$ on $\mathscr{C}$ if, for any $\succ', \succ'' \in \mathcal{L}(\mathscr{C})/\{\succ\}$ such that $\trianglerighteq_{\succ} \succ''$, it holds that

$$\mathbf{n}_{\mathscr{R}}(\succ) > \mathbf{n}_{\mathscr{R}}(\succ') \geq \mathbf{n}_{\mathscr{R}}(\succ'').$$

In the following example, the notion of monotonicity of a profile of rankings is illustrated.

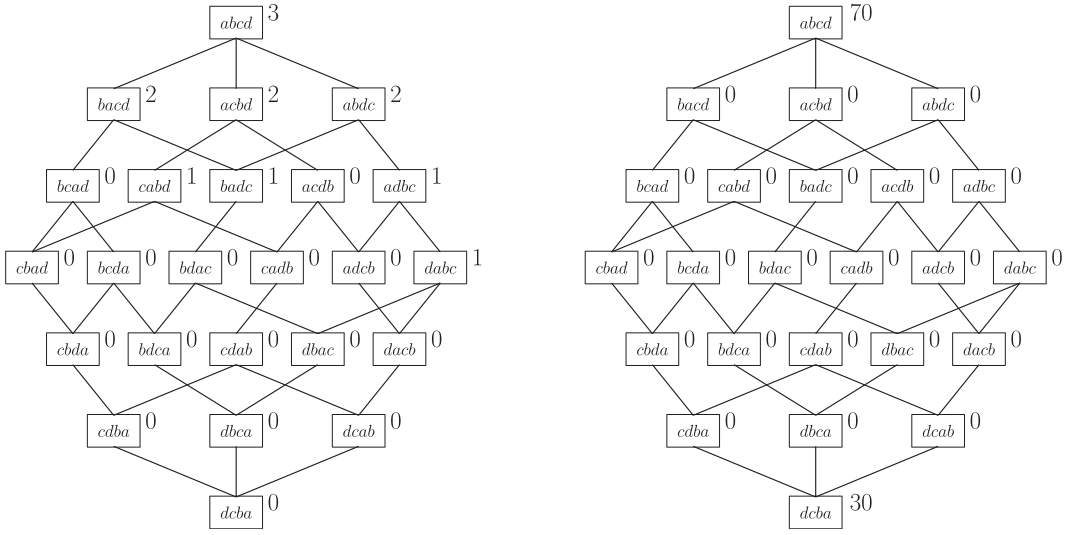

FIGURE 6 Profile of rankings in Table 1 (left) and profile of rankings in Table 3 (right) represented on the Hasse diagram of $\trianglerighteq_{\succ}$ for the ranking $a \succ b \succ c \succ d$

Example 4. Let $\mathscr{C} = \{a, b, c, d\}$ be a set of candidates and $\mathscr{R}$ be the profile of rankings given in Example 1. In the left side of Figure 6 the profile of rankings is represented on the Hasse diagram of $\trianglerighteq_{\succ}$ for the ranking $a \succ b \succ c \succ d$. Note that the values decrease when going from top to bottom in the Hasse diagram of $\trianglerighteq_{\succ}$. Therefore, the profile of rankings given in this example is monotone w.r.t. the ranking $a \succ b \succ c \succ d$.

Unlike (recursive) monotonicity of the scorix and monotonicity of the votrix, monotonicity of the profile of rankings w.r.t. a ranking does not result in an agreement of a family of ranking rules. Rather, as discussed in Pérez-Fernández et al,[18] monotonicity of the profile of rankings is linked to the real existence of a true ranking on the set of candidates. This complies with the philosophy advocated by Rousseau[25] and Condorcet,[9] where personal preferences are not considered and the goal is to identify the 'general will.' This philosophy is clearly described by Arrow[1]: "each individual has two orderings, one which governs him in his everyday actions, and one which would be relevant under some ideal conditions and which is in some sense truer than the first ordering. It is the latter which is considered relevant to social choice, and it is assumed that there is complete unanimity with regard to the truer individual ordering." From this reflection, one could conclude that there are two different settings for the aggregation of rankings: there exists a latent true ranking that voters try to identify, the goal of the aggregation being to identify said true ranking, or, contrarily, voters have conflicting opinions, the goal of the aggregation being to agree on a compromise ranking. In Pérez-Fernández et al,[18] a statistical test for testing the existence of a latent true ranking based on the notion of monotonicity of a profile of rankings is described. In the following, we provide an example of a profile of rankings for which its aggregation should be considered as a search for a compromise ranking.

Example 5. Let $\mathscr{C} = \{a, b, c, d\}$ be a set of candidates and $\mathscr{R}$ be the profile of $r = 100$ rankings given in Table 3.

TABLE 3 Profile $\mathscr{R}$ of $r = 100$ rankings on $\mathscr{C} = \{a, b, c, d\}$

Frequency	Rankings on $\mathscr{C}$
70	$a \succ b \succ c \succ d$
30	$d \succ c \succ b \succ a$

In the right side of Figure 6 the profile of rankings is represented on the Hasse diagram of $\trianglerighteq_{\succ}$ for the ranking $a \succ b \succ c \succ d$. Note that the values do not decrease when going from top to bottom in the Hasse diagram of $\trianglerighteq_{\succ}$. Therefore, the profile of rankings given in this example is not (close to being) monotone w.r.t. the ranking $a \succ b \succ c \succ d$ (or w.r.t. any other ranking on $\mathscr{C}$). This hints that, when aggregating the rankings in this profile, we are seeking for a compromise solution rather than a true ranking.

Like in the case of the scorix, it seems intuitive to extend the property of monotonicity of a profile of rankings to all possible restrictions of this profile of rankings.

Definition 11. Let $\mathscr{C}$ be a set of k candidates and r be the number of voters. A profile $\mathscr{R}$ of r rankings on $\mathscr{C}$ is said to be (strictly) recursively monotone w.r.t. a ranking $\succ$ on $\mathscr{C}$ if, for any nonempty subset $\mathscr{C}' \subseteq \mathscr{C}$, the restriction of $\mathscr{R}$ to $\mathscr{C}'$ is (strictly) monotone w.r.t. the restriction of $\succ$ to $\mathscr{C}'$.

In the following example, the notion of recursive monotonicity of a profile of rankings is illustrated.

Example 6. Let $\mathscr{C} = \{a, b, c, d\}$ be a set of candidates and $\mathscr{R}$ be the profile of rankings given in Example 1 In Figure 7 the profile of rankings and all its restrictions are represented on the Hasse diagram of $\trianglerighteq_{\succ}$ and each $\trianglerighteq_{\succ'}$ corresponding to each restriction $\succ'$ of $\succ$ to each subset of $\mathscr{C}$ (of cardinality greater than or equal to two) for the ranking $a \succ b \succ c \succ d$. Note that the profile of rankings and all its restrictions are monotone w.r.t. the corresponding restriction of $a \succ b \succ c \succ d$. Therefore, the profile of rankings given in this example is recursively monotone w.r.t. the ranking $a \succ b \succ c \succ d$.

Recursive monotonicity of a profile of rankings obviously is a stronger property than monotonicity of a profile of rankings* and a weaker property than unanimity. Furthermore, it guarantees the existence of (and compliance with) the Condorcet ranking, although it does not guarantee the existence of (nor compliance with) the Borda ranking†.

Theorem 3. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings on $\mathscr{C}$ given by the voters and $\succ$ be a ranking on $\mathscr{C}$. The following statements hold:*

*Recursive monotonicity of a profile of rankings obviously implies its monotonicity. However, the converse is not true. Consider the set $\mathscr{C} = \{a, b, c, d\}$ of $k = 4$ candidates and the profile of $r = 13$ rankings where five voters express the ranking $a \succ b \succ c \succ d$, four voters express the ranking $a \succ c \succ b \succ d$ and four voters express the ranking $a \succ c \succ d \succ b$. This profile of rankings is monotone w.r.t. $a \succ b \succ c \succ d$, but its corresponding restriction to $\mathscr{C}' = \{a, b, c\}$ is not monotone w.r.t. $a \succ b \succ c$.

†Consider the set $\mathscr{C} = \{a, b, c\}$ of $k = 3$ candidates and the profile of $r = 9$ rankings where five voters express the ranking $a \succ b \succ c$, two voters express the ranking $b \succ a \succ c$ and two voters express the ranking $b \succ c \succ a$. This profile of rankings is recursively monotone w.r.t. $a \succ b \succ c$, whereas the Borda ranking is $b \succ a \succ c$.

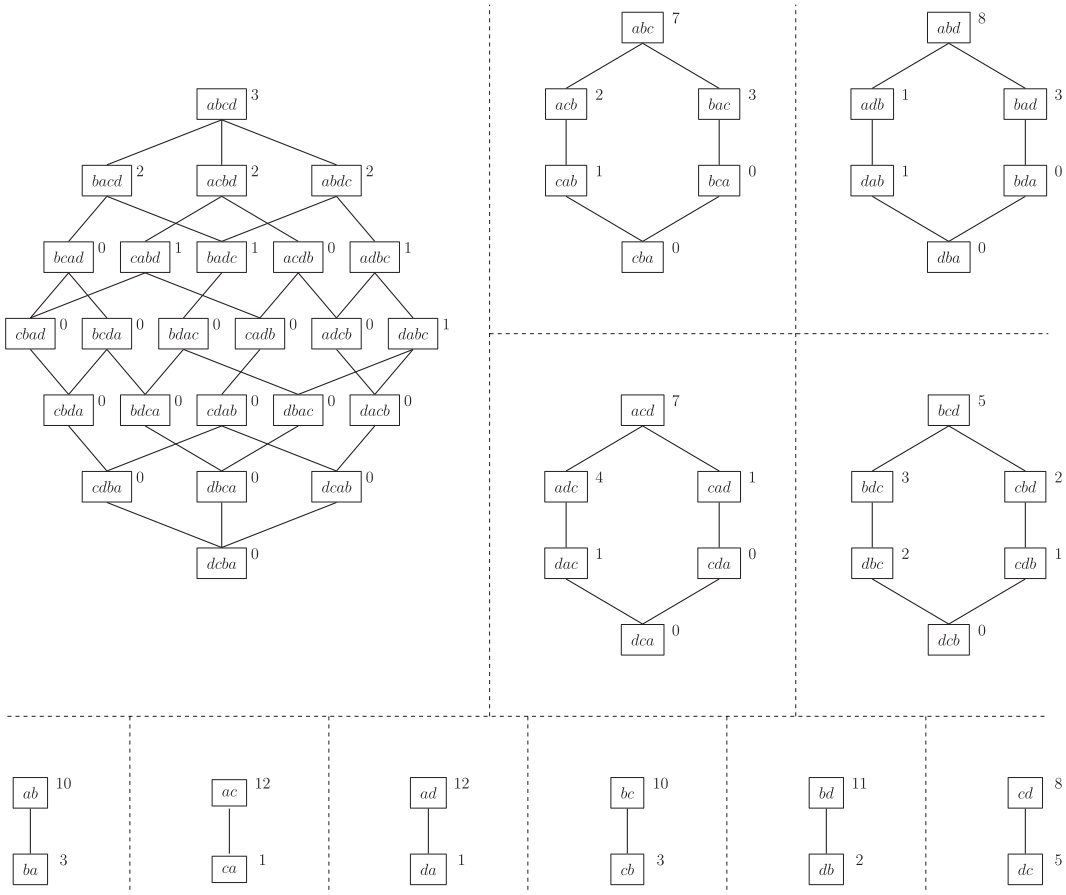

FIGURE 7 Profile of rankings and restrictions of the profile of rankings in Table 1 represented on the Hasse diagrams of $\trianglerighteq_{\succ}$ and each $\trianglerighteq_{\succ}{}'$ corresponding to each restriction $\succ'$ of $\succ$ to each subset of $\mathscr{C} = \{a, b, c, d\}$ for the ranking $a \succ b \succ c \succ d$

- **(i)** *If $\mathscr{R}$ is the unanimous profile of rankings where every voter expresses $\succ$, then $\mathscr{R}$ is recursively monotone w.r.t. $\succ$.*
- **(ii)** *If $\mathscr{R}$ is recursively monotone w.r.t. $\succ$, then $\mathscr{R}$ is monotone w.r.t. $\succ$.*
- **(iii)** *If $\mathscr{R}$ is recursively monotone w.r.t. $\succ$, then $\mathscr{R}$ is the Condorcet ranking.*

3 | ACCLAMATION

Some works have addressed the computation of the probability of the agreement between different ranking rules. For instance, for the case of three-candidate elections, Gehrlein and Lepelley[26] computed the probability that both the Condorcet winner and the Borda winner coincide and Merlin et al[27] computed the probability of all Condorcet procedures, all scoring rules and all runoff methods resulting in the same outcome. Unfortunately, the computation of these probabilities becomes a difficult task for a (moderately) large number of voters.

Here, the aim is not to compute the probability of the agreement between different ranking rules, but rather to analyse the conditions under which determining the winning ranking on the set of candidates is obvious. Unanimity obviously is one of this situations and, unfortunately, it is the only situation under which the winning ranking on the set of candidates is indisputably determined. Fortunately, as discussed in the previous section, monotonicity of different representations of votes can be understood as a cornerstone of social choice theory where almost all ranking rules lead to the same social outcome.

First, recursive monotonicity of the scorix assures that all ranking rules based on positional information lead to the same ranking on the set of candidates. Second, (recursive) monotonicity of the votrix assures that all ranking rules based on pairwise information lead to the same ranking on the set of candidates. Third, recursive monotonicity of the profile of rankings assures that the result of the ranking rule is not a compromise solution.

In this section, we propose to jointly consider these three types of (recursive) monotonicity in order to define a weaker condition than unanimity, but that still leads to an obvious social outcome. From now on, a ranking w.r.t. which the scorix is recursively monotone, the votrix is (recursively) monotone and the profile of rankings is recursively monotone, is referred to as an acclaimed ranking*.

Definition 12. Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings on $\mathscr{C}$ given by the voters, S be the scorix induced by $\mathscr{R}$ and V be the votrix induced by $\mathscr{R}$. A ranking $\succ$ on $\mathscr{C}$ is called the acclaimed ranking for $\mathscr{R}$ if the following three statements hold:

(i) S is recursively monotone w.r.t. $\succ$.
(ii) V is monotone w.r.t. $\succ$.
(iii) $\mathscr{R}$ is recursively monotone w.r.t. $\succ$.

In case there exists an acclaimed ranking for a given profile of rankings, we say that the profile of rankings belongs to the acclamation consensus state.

By definition, acclamation is a weaker property than unanimity and, obviously, a stronger property than (recursive) monotonicity of the scorix, monotonicity of the votrix and (recursive) monotonicity of the profile of rankings.

Theorem 4. *Let $\mathscr{C}$ be a set of k candidates, $\mathscr{R}$ be the profile of r rankings on $\mathscr{C}$ given by the voters, S be the scorix induced by $\mathscr{R}$, V be the votrix induced by $\mathscr{R}$ and $\succ$ be a ranking on $\mathscr{C}$. The following statements hold:*

(i) *If $\mathscr{R}$ is the unanimous profile of rankings where every voter expresses $\succ$, then $\succ$ is the acclaimed ranking for $\mathscr{R}$.*
(ii) *If $\succ$ is the acclaimed ranking for $\mathscr{R}$, then S is (recursively) monotone w.r.t. $\succ$.*
(iii) *If $\succ$ is the acclaimed ranking for $\mathscr{R}$, then V is monotone w.r.t. $\succ$.*
(iv) *If $\succ$ is the acclaimed ranking for $\mathscr{R}$, then $\mathscr{R}$ is (recursively) monotone w.r.t. $\succ$.*
(v) *If $\succ$ is the acclaimed ranking for $\mathscr{R}$, then $\succ$ is the Condorcet ranking.*
(vi) *If $\succ$ is the acclaimed ranking for $\mathscr{R}$, then $\succ$ is the Borda ranking.*

*The term acclamation historically refers to a voting system used in Ancient Greece, where the winning candidate was decided by the (loudest) shouts of the people.[7]

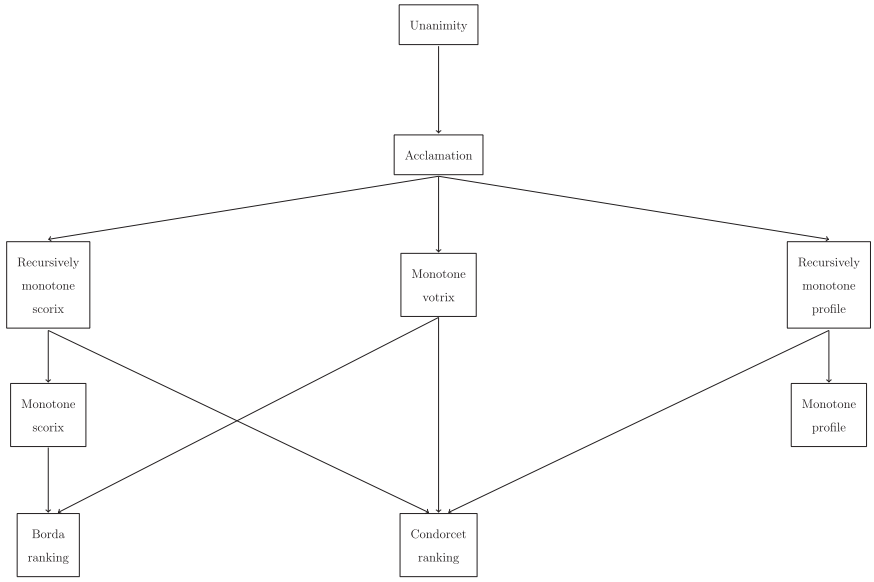

FIGURE 8 Relations between the different properties

Corollary 1. *Let $\mathscr{C}$ be a set of k candidates and $\mathscr{R}$ be the profile of r rankings on $\mathscr{C}$ given by the voters. If there exists an acclaimed ranking for $\mathscr{R}$, then it is unique.*

Figure 8 illustrates the relations between the different types of properties discussed here. In this figure, an arrow indicates that the property from which the arrow starts implies the property to which the arrow points.

As illustrated in Table 4 in case $\succ$ is the acclaimed ranking for a given profile of rankings, the ranking $\succ$ is a winning ranking and/or the first ranked candidate in $\succ$ is a winning candidate for the most prominent voting rules. In,Table 4 a symbol ✓ (resp. $^-$) in the column WC means that the first ranked candidate in the acclaimed ranking is (resp. does not need to be) a *Winning Candidate* for the method corresponding to the row; a symbol ✓ (resp. $^-$) in the column UWC means that the first ranked candidate in the acclaimed ranking is (resp. does not need to be) the *Unique Winning Candidate* for the method corresponding to the row; a symbol ✓ (resp. $^-$) in the column WR means that the acclaimed ranking is (resp. does not need to be) a *Winning Ranking* for the method corresponding to the row; and a symbol ✓ (resp. $^-$) in the column UWR means that the acclaimed ranking is (resp. does not need to be) the *Unique Winning Ranking* for the method corresponding to the row. The symbol * means that the method corresponding to the row is not explicitly defined for identifying a winning ranking. We refer to the Appendix of this article for a formal proof of these results.

4 | THE RANKING RULE

4.1 | Definition

Several authors, such as Nitzan,[28] Lerer and Nitzan,[29] Campbell and Nitzan,[30] Meskanen and Nurmi,[11,31] Andjiga et al[12], and Elkind et al[13] have advocated that (most) ranking rules can be characterized as minimizing the distance to a consensus state for some appropriate metric (for a

TABLE 4 Concordance with the notion of acclamation by the most prominent methods in social choice theory

Method	WC	UWC	WR	UWR
Plurality	✓	–	✓	–
Borda count	✓	✓	✓	✓
Veto	✓	–	✓	–
Best-worst voting systems	✓	–	✓	–
Scoring (ranking) rules	✓	–	✓	–
Elimination methods based on a scoring (ranking) rule	✓	–	✓	–
Simple majority rule	✓	✓	✓	✓
Dodgson	✓	✓	*	*
Condorcet's least-reversals	✓	✓	*	*
Kemeny	✓	✓	✓	✓
Copeland	✓	✓	✓	✓
Tideman	✓	✓	✓	✓
Schulze	✓	✓	✓	✓
Simpson	✓	✓	✓	–
Bucklin	✓	–	*	*

discussion on classical metrics for rankings, we refer to Cook[32]). This characterization is known as metric rationalisation of ranking rules.

However, when we are not dealing with a notion of closeness in the most geometrical sense, the symmetry axiom of a metric might not be necessary. Quoting Tversky and Gati on the study of similarity measures,[33,34] one can understand that the term closeness is not always interpreted as a symmetric term: "The poet writes 'my love is as deep as the ocean,' not 'the ocean is as deep as my love,' because the ocean epitomizes depth." Even more importantly, the triangle inequality of a metric may not always be linked to a notion of closeness. For instance, when thinking of a human, a centaur and a horse, the term closeness is not related with the triangle inequality (the perceived distance of a human to a horse exceeds the perceived distance of a human to a centaur, plus that of a centaur to a horse). Nevertheless, there is a clear betweenness relation: a centaur is between a human and a horse and, therefore, a human should always be closer to a centaur than to a horse.

Closeness is a vague term here. From a geometrical point of view, symmetry and the triangle inequality are needed. Nevertheless, in the rationalisation of ranking rules, closeness is not defined by a geometrical concept. Here, this closeness is related to the notion of (local) penalty function used in the aggregation of real numbers,[6] where the axioms of symmetry and the triangle inequality are no longer required, but an additional axiom providing the penalty with a well-founded semantic basis is required. In this work, this well-founded semantic basis is captured by requiring the preservation of a natural betweenness relation. Therefore, closeness is no longer measured by a metric, but by a monometric*.

*Note that every metric is a monometric w.r.t. a certain betweenness relation.[14]

Definition 13 (Pérez-Fernández et al[14]). Let $\mathscr{C}$ be a set of k candidates and r be the number of voters. A function $M\colon \mathcal{L}(\mathscr{C})^r \times \mathcal{L}(\mathscr{C})^r \to \mathbb{R}$ is called a monometric (w.r.t. the betweenness relation introduced by Kemeny[2]) if it satisfies the following three properties:

(i) Non-negativity: for any $\mathscr{R}, \mathscr{R}' \in \mathscr{L}(\mathscr{C})^r$, it holds that $M(\mathscr{R}, \mathscr{R}') \geq 0$.
(ii) Coincidence: for any $\mathscr{R}, \mathscr{R}' \in \mathscr{L}(\mathscr{C})^r$, it holds that $M(\mathscr{R}, \mathscr{R}') = 0 \Leftrightarrow \mathscr{R} = \mathscr{R}'$.
(iii) Compatibility: for any $\mathscr{R}, \mathscr{R}', \mathscr{R}'' \in \mathscr{L}(\mathscr{C})^r$ such that

$$\sum_{j=1}^{r} K(\succ_j, \succ''_j) = \sum_{j=1}^{r} K(\succ_j, \succ'_j) + \sum_{j=1}^{r} K(\succ'_j, \succ''_j),$$

where K denotes the Kendall metric[35] between rankings*, it holds that

$$M(\mathscr{R}, \mathscr{R}') \leq M(\mathscr{R}, \mathscr{R}'').$$

Due to all aforementioned reasons, it was advocated by Pérez-Fernández et al[14] that monometrics (instead of metrics) should be considered in the rationalisation of ranking rules, leading to the introduction of the monometric rationalisation of ranking rules. This direction was further extended in Pérez-Fernández et al[36] by introducing the search for consensus states in the case of rankings with ties.

We recall that the introduction of consensus states broader than unanimity, but that still lead to an obvious ranking on the set of candidates, represents a valuable topic in the field of social choice theory because the broader the consensus state, the less the choice of (mono) metric matters. This is due to the fact that the profile of rankings is typically not close to belonging to the unanimity consensus state (thus, the choice of (mono)metric plays a key role), whereas the profile of rankings is always closer to belonging to a broader consensus state than unanimity (thus, the choice of (mono)metric plays a lesser role). For this reason, we advocate for the use of acclamation, which is the broadest consensus state for which we could not identify a prominent ranking rule disagreeing with the associated winning ranking (as shown in Table 4).

Similarly to the method of Kemeny where we compute the Kemeny score for each ranking $\succ$, here we have the corresponding cost associated with a closest profile of rankings for which $\succ$ is the acclaimed ranking (measured by means of the chosen monometric $M\colon \mathcal{L}(\mathscr{C})^r \times \mathcal{L}(\mathscr{C})^r \to \mathbb{R}$). Formally, given the profile $\mathscr{R}$ of r rankings on $\mathscr{C}$ given by the voters, we have the following cost for each ranking $\succ$:

$$C_M(\succ) = \min_{\substack{\mathscr{R}' \in \mathcal{L}(\mathscr{C})^r \text{s.t.} \\ \succ \text{ is acclaimed ranking}}} M(\mathscr{R}, \mathscr{R}').$$

In particular, we propose to consider the monometric $\mathbb{K}\colon \mathscr{L}(\mathscr{C})^r \times \mathscr{L}(\mathscr{C})^r \to \mathbb{R}$ defined by the sum of Kendall distances (as proposed by Pérez-Fernández et al[14]), that is, $\mathbb{K}(\mathscr{R}, \mathscr{R}') = \sum_{i=1}^{r} K(\succ_i, \succ_i')$ for any $\mathscr{R}, \mathscr{R}' \in \mathcal{L}(\mathscr{C})^r$. This leads to the introduction of a Kemeny-like method where instead of unanimity we search for acclamation.

*The Kendall metric K is defined as, $K(\succ_1, \succ_2) = \#\{(a_{i_1}, a_{i_2}) \in \mathscr{C}^2 | a_{i_1} \succ_1 a_{i_2}, a_{i_2} \succ_2 a_{i_1}\}$, for any two rankings $\succ_1$ and $\succ_2$.

TABLE 5 Profile $\mathscr{R}$ of $r = 101$ rankings on $\mathscr{C} = \{a, b, c, d\}$

Frequency	Rankings on $\mathscr{C}$
51	$a \succ b \succ c \succ d$
50	$b \succ c \succ d \succ a$

Thus, we advocate that a winning ranking should be one whose corresponding closest profile of rankings for which $\succ$ is the acclaimed ranking is the closest to the profile of rankings given by the voters:

$$\arg\min_{\succ \in \mathcal{L}(\mathscr{C})} C_{\mathbb{K}}(\succ).$$

Obviously, like in almost all methods for the aggregation of rankings, the winning ranking does not need to be unique.

4.2 | Correspondence with intuition

Although the Condorcet ranking is a popular notion among social choice theorists, it has also withstood some criticism. For instance, see the following quote by Saari[37]: "the combination of the pairwise vote with the Condorcet terms loses the crucial fact that voters have transitive preferences. [...] An equally surprising assertion is that rather than being the standard, the Condorcet winner must be held suspect." In most situations, the Condorcet ranking is indeed the most natural winning ranking. However, one could find some examples in which the Condorcet ranking is at the very least an arguable winner.

Consider the set $\mathscr{C} = \{a, b, c, d\}$ of $k = 4$ candidates and the profile $\mathscr{R}$ of $r = 101$ rankings on $\mathscr{C}$ given in Table 5 For this profile of rankings, there exists a Condorcet winner (candidate a) and a Condorcet ranking (the ranking $a \succ b \succ c \succ d$).

In case we apply the ranking rule introduced in this paper based on the search for acclamation, we obtain the ranking $b \succ a \succ c \succ d$ (Table 6), which seems to be a more intuitive winning ranking for this profile of rankings than the ranking $a \succ b \succ c \succ d$ obtained by the method of Kemeny.

4.3 | Analysis of the properties

Obviously, simple axioms, such as non-dictatorship*, non-imposition†, anonymity‡ and neutrality§, are trivially satisfied by the ranking rule.

The profile of rankings in Table 5 also implies the failure of several well-known properties for ranking rules. First, as candidate a is the Condorcet winner and is chosen the winner by more than half of the voters while not being the winner for our proposed ranking rule, we conclude that the properties of Condorcet consistency¶ and the majority criterion# are not satisfied.

*A ranking rule is said to satisfy the non-dictatorship criterion if there is no voter whose ranking is always elected the winner.

†A ranking rule is said to satisfy the non-imposition criterion if every ranking is selected as the winner in case it is unanimously decided by the voters.

‡A ranking rule is said to satisfy the anonymity criterion if reassigning the rankings over the voters does not change the outcome.

§A ranking rule is said to satisfy the neutrality criterion if some permutation of the candidates is applied to each voters' ranking, the winner is the result of this same permutation.

¶A ranking rule is said to be Condorcet consistent if the first ranked candidate by the ranking rule always coincides with the Condorcet winner in case the latter exists.

TABLE 6 Values $C_{\mathbb{K}}(\succ)$ for each ranking $\succ$ on $\mathscr{C} = \{a, b, c, d\}$ given the profile of rankings in Table 5

Ranking ($\succ$)	$C_{\mathbb{K}}(\succ)$	Ranking ($\succ$)	$C_{\mathbb{K}}(\succ)$	Ranking ($\succ$)	$C_{\mathbb{K}}(\succ)$	Ranking ($\succ$)	$C_{\mathbb{K}}(\succ)$
$a \succ b \succ c \succ d$	100	$b \succ a \succ c \succ d$	85	$c \succ a \succ b \succ d$	132	$d \succ a \succ b \succ c$	203
$a \succ b \succ d \succ c$	151	$b \succ a \succ d \succ c$	104	$c \succ a \succ d \succ b$	191	$d \succ a \succ c \succ b$	210
$a \succ c \succ b \succ d$	127	$b \succ c \succ a \succ d$	86	$c \succ b \succ a \succ d$	105	$d \succ b \succ a \succ c$	192
$a \succ c \succ d \succ b$	202	$b \succ c \succ d \succ a$	103	$c \succ b \succ d \succ a$	154	$d \succ b \succ c \succ a$	203
$a \succ d \succ b \succ c$	202	$b \succ d \succ a \succ c$	133	$c \succ d \succ a \succ b$	204	$d \succ c \succ a \succ b$	211
$a \succ d \succ c \succ b$	205	$b \succ d \succ c \succ a$	128	$c \succ d \succ b \succ a$	203	$d \succ c \succ b \succ a$	206

TABLE 7 Profile $\mathscr{R}$ of $r = 101$ rankings on $\mathscr{C} = \{a, b, c, d\}$

Freq.	Rankings on $\mathscr{C}$
51	$a \succ b \succ c \succ d$
50	$b \succ a \succ c \succ d$

Unfortunately, this ranking rule is not independent of clones*, as $S = \{b, c, d\}$ clearly is a set of clones, but b is no longer the winner in case c and d are eliminated from the ballot (contradicting the first condition for the independence of clones).

The axiom of independence of irrelevant alternatives† is not satisfied either. Indeed, for the profile of rankings in Table 7 candidate a is ranked at a better position than candidate b in the winning ranking ($a \succ b \succ c \succ d$), whereas for the profile of rankings in,Table 5 candidate b is ranked at a better position than candidate a in the winning ranking ($b \succ a \succ c \succ d$). Note that the relative order of candidates a and b has not been altered, while their order in both winning rankings has.

Moreover, the proposed ranking rule is not homogeneous‡. For instance, consider the set $\mathscr{C} = \{a, b, c, d\}$ of $k = 4$ candidates and the profile $\mathscr{R}$ of $r = 10$ rankings given in Table 8

According to the ranking rule introduced in this paper, there are six winning ranking for this profile of rankings ($a \succ b \succ c \succ d$, $a \succ c \succ b \succ d$, $b \succ a \succ c \succ d$, $b \succ c \succ a \succ d$, $c \succ a \succ b \succ d$ and $c \succ b \succ a \succ d$). However, in case we repeat the rankings of the voters twice, the winning ranking is solely the ranking $a \succ c \succ b \succ d$.

In addition, due to the fact that a closest profile of rankings for which each ranking on $\mathscr{C}$ is the acclaimed ranking needs to be obtained, the computation of the winning ranking for this ranking rule is a difficult problem. As discussed in Bartholdi et al,[39] this is a common issue with many voting schemes.

#A ranking rule is said to satisfy the majority criterion if the first ranked candidate by the ranking rule always coincides with a candidate that is ranked first by more than half of the voters in case the latter exists.

*According to Tideman,[38] "a proper subset of two or more candidates, S, is a set of clones if no voter ranks any candidate outside of S as either tied with any element of S or between any two elements of S. [...] A ranking rule is said to be independent of clones if and only if the following two conditions are met when clones are on the ballot: 1. A candidate that is a member of a set of clones wins if and only if some member of that set of clones wins after a member of the set is eliminated from the ballot. 2. A candidate that is not a member of a set of clones wins if and only if that candidate wins after any clone is eliminated from the ballot."

†A ranking rule is said to be independent of irrelevant alternatives if the order between two alternatives x and y depends only on the relative positions of x and y in the rankings given by the voters.

‡A ranking rule is said to be homogeneous if a winner remains the same in case the rankings of the voters are repeated a finite number of times.

TABLE 8 Profile $\mathscr{R}$ of $r = 10$ rankings on $\mathscr{C} = \{a, b, c, d\}$

Freq.	Rankings on $\mathscr{C}$
4	$b \succ a \succ d \succ c$
2	$c \succ a \succ b \succ d$
1	$a \succ c \succ d \succ b$
1	$b \succ c \succ a \succ d$
1	$c \succ a \succ d \succ b$
1	$d \succ c \succ b \succ a$

The reader may note that (unsurprisingly) all the previously discussed non-satisfied properties (except for the above-discussed Condorcet consistency and homogeneity) are also not satisfied by the method of Kemeny,[2] probably the best-known ranking rule. Actually, the ranking rule proposed in Section 4 would satisfy the property of homogeneity in case we would consider a slight variation of the Kendall metric,[35] as discussed by Fishburn[40] for the method of Dodgson.[8]

5 | CONCLUSIONS AND OPEN PROBLEMS

In this article, we have analysed a new consensus state, acclamation, that serves as a meeting point for the most prominent ranking rules in social choice theory. Acclamation results in a natural condition under which methods based on either positional or pairwise information lead to the same social outcome. In particular, the acclaimed ranking is a natural sufficient condition for the Borda ranking and the Condorcet ranking to exist and coincide. Moreover, a Kemeny-like ranking rule based on the search for acclamation has been introduced, resulting in an intuitively appealing winning ranking. Like unanimity, acclamation is a natural consensus state leading to an obvious winning ranking, and the search for acclamation does not rely on the choice of the Kendall metric as strongly as the search for unanimity.

For computing the winning ranking of this ranking rule, we are currently solving the cumbersome Integer Linear Programming problem discussed in.[14] As mentioned in Section 4.3 computing the winning ranking is a difficult problem. However, there is still a lot of room for improvement in the computational implementation of this method. We highlight the introduction of effective pruning techniques reducing the number of rankings for which we need to compute the cost associated with a closest profile of rankings for which $\succ$ is the acclaimed ranking. Another reasonable possibility would be to adopt a heuristic approach[41,42] that would speed up the computation time of our proposed ranking rule, although some voters could feel deceived if a (slighlty) different outcome than the optimal one would be obtained.

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APPENDIX

In this Appendix, we prove the results illustrated in Table 4.

For plurality,[43] the Borda count,[3] veto,[44] best-worst voting systems,[45] scoring (ranking) rules,[46] and elimination methods based on a scoring (ranking) rule,[15] the facts that the first ranked candidate in the acclaimed ranking is a winning candidate and that the acclaimed ranking is a winning ranking are a direct result from Theorem 1. This theorem also implies that the first ranked candidate in the acclaimed ranking is the unique winning candidate for the Borda count and that the acclaimed ranking is the unique winning ranking for the Borda count.

TABLE A1 Profile $\mathscr{R}$ of $r = 28$ rankings on $\mathscr{C} = \{a, b, c, d\}$

Ranking	Frequency	Ranking	Freq.	Ranking	Frequency	Ranking	Frequency
$a \succ b \succ c \succ d$	4	$b \succ a \succ c \succ d$	3	$c \succ a \succ b \succ d$	1	$d \succ a \succ b \succ c$	1
$a \succ b \succ d \succ c$	3	$b \succ a \succ d \succ c$	2	$c \succ a \succ d \succ b$	1	$d \succ a \succ c \succ b$	1
$a \succ c \succ b \succ d$	1	$b \succ c \succ a \succ d$	2	$c \succ b \succ a \succ d$	1	$d \succ b \succ a \succ c$	1
$a \succ c \succ d \succ b$	1	$b \succ c \succ d \succ a$	1	$c \succ b \succ d \succ a$	0	$d \succ b \succ c \succ a$	0
$a \succ d \succ b \succ c$	1	$b \succ d \succ a \succ c$	2	$c \succ d \succ a \succ b$	0	$d \succ c \succ a \succ b$	0
$a \succ d \succ c \succ b$	1	$b \succ d \succ c \succ a$	1	$c \succ d \succ b \succ a$	0	$d \succ c \succ b \succ a$	0

We prove that the uniqueness is not assured for the remaining ranking rules mentioned in this paragraph by providing a counterexample.

Consider the set $\mathscr{C} = \{a, b, c, d\}$ of $k = 4$ candidates and the profile $\mathscr{R}$ of $r = 28$ rankings given in Table A1.

The scorix induced by $\mathscr{R}$ is:

$$S = \begin{pmatrix} 11 & 9 & 6 & 2 \\ 11 & 9 & 4 & 4 \\ 3 & 5 & 10 & 10 \\ 3 & 5 & 8 & 12 \end{pmatrix}.$$

Although $a \succ b \succ c \succ d$ is the acclaimed ranking for $\mathscr{R}$, all rankings $a \succ b \succ c \succ d$, $a \succ b \succ d \succ c$, $b \succ a \succ c \succ d$ and $b \succ a \succ d \succ c$ are winning rankings for the plurality rule. Analogously, in case we reverse the order of the candidates in all rankings in the profile of rankings, we obtain a profile $\mathscr{R}'$ of rankings where, although $d \succ c \succ b \succ a$ is the acclaimed ranking for $\mathscr{R}'$, all rankings $c \succ d \succ a \succ b$, $c \succ d \succ b \succ a$, $d \succ c \succ a \succ b$ and $d \succ c \succ b \succ a$ are winning rankings for the veto rule. As best-worst voting systems, scoring (ranking) rules and elimination methods based on a scoring (ranking) rule have both the plurality and the veto rules as a particular case, we conclude that the first ranked candidate in the acclaimed ranking does not need to be the unique winner, and the acclaimed ranking does not need to be the unique winning ranking for any of the aforementioned ranking rules.

By definition of the simple majority rule,[47] in case of existence of the Condorcet ranking, which is assured to coincide with the acclaimed ranking in case the latter exists, the (unique) winning candidate coincides with the first ranked candidate in the Condorcet ranking and the (unique) winning ranking coincides with the Condorcet ranking. Similarly, as both the method of Dodgson[8] and Condorcet's least-reversals method[11] are based on the search for the candidate that is the closest to becoming the Condorcet winner, in case of existence of the Condorcet winner, the (unique) winning candidate for both methods coincides with the first ranked candidate in the Condorcet ranking.

The methods of Kemeny,[2] Copeland,[48] Tideman,[38] Schulze,[49] and Simpson[50,51] are Condorcet methods, that is, in case of existence of a Condorcet winner, they select this Condorcet winner as the unique winning candidate. Moreover, the first four are additionally Condorcet ranking methods, that is, in case of existence of a Condorcet ranking, they select this Condorcet ranking as the unique winning ranking. Due to the property of monotonicity of the votrix, the method of Simpson, which ranks the candidates according to their greatest pairwise

TABLE A2 Profile $\mathscr{R}$ of $r = 11$ rankings on $\mathscr{C} = \{a, b, c, d\}$

Frequency	Rankings on $\mathscr{C}$
6	$a \succ b \succ c \succ d$
5	$b \succ a \succ c \succ d$

defeat, is trivially assured to select the acclaimed ranking as the winning ranking. However, the uniqueness is not assured in this case.

Consider the set $\mathscr{C} = \{a, b, c, d\}$ of $k = 4$ candidates and the profile $\mathscr{R}$ of $r = 11$ rankings given in Table A2.

The votrix induced by $\mathscr{R}$ is:

$$V = \begin{pmatrix} 0 & 6 & 11 & 11 \\ 5 & 0 & 11 & 11 \\ 0 & 0 & 0 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that candidates c and d have the same greatest pairwise defeat. Therefore, although $a \succ b \succ c \succ d$ is the acclaimed ranking for $\mathscr{R}$, both rankings $a \succ b \succ c \succ d$ and $a \succ b \succ d \succ c$ are winning rankings for the method of Simpson.

As the vector of positions of candidate a (strictly) dominates the vectors of positions of all other candidates, we conclude that the first ranked candidate in the acclaimed ranking is a winning candidate for the method of Bucklin,[52] For proving that the uniqueness does not hold, consider again the profile $\mathscr{R}$ of $r = 28$ rankings given in Table A1. Note that there is no candidate that is ranked at the first position by more than half of the number of voters. Therefore, according to the method of Bucklin, we need to consider also the number of times that each candidate is ranked at the second position. Now, candidates a and b are ranked at the first or second position by 20 voters, which is more than half of the number of voters. Therefore, candidates a and b are winning candidates for the method of Bucklin, while candidate b is not the first ranked candidate in the acclaimed ranking.